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A Weierstrass Approximation Theorem for Topological Vector Spaces

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P. M. Prenter [1, pp. 341–351] showed that if H is a real separable Hilbert space and K is a compact subset of H, then the polynomial operators of finite rank are dense in $C(K; H)$ in the uniform norm. Later as a consequence of more general considerations, Prolla and Machado [2, pp. 247–258] proved that if E and F are real locally convex Hausdorff spaces, then the polynomial operators of finite type are dense in $C(E; F)$ equipped with the compact-open topology. In this paper we modify Prenter's approach to obtain a Weierstrass theorem when E and F are *not necessarily convex.*

Before stating the main theorem let us make the following definition: A sequence of linear operators (t_n) on a topological vector space has the *Grothendieck approximation propety* if and only if (1) each t_n has finite rank, (2) the sequence (t_n) is equicontinuous, and (3) the sequence (t_n) converges uniformly on compact subsets to the identity operator of the space [3, pp. 108–115]. For example, in a complete linear metric space with a Schauder basis (e_n), the sequence of projections

$$t_n: \sum_{k=1}^{\infty} a_k e_k \mapsto \sum_{k=1}^{n} a_k e_k$$

has the Grothendieck approximation property [3, p. 115].

THEOREM. *Let E and F be real Hausdorff topological vector spaces. Suppose E has a sequence (s_n) of projections with the Grothendieck approximation property and suppose F has a sequence (t_n) with the Grothendieck approximation property. Then the polynomials from E into F of finite rank are dense in $C(E; F)$ in the compact-open topology.*

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As we have noted above, the theorem holds for complete linear metric spaces with Schauder bases. For example, let (p_n) be a sequence of real numbers satisfying $0 < p_n \leqslant 1$ and define

$$l(p_n) = \left\{ x \in s\colon \sum_{n=1}^{\infty} |x_n|^{p_n} < \infty \right\}$$

(s is the vector space of all real sequences). Then with the natural metric

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^{p_n}$$

$l(p_n)$ is a complete linear metric space with a Schauder basis (4). The spaces $l(p_n)$ (for different real sequences (p_n), $0 < p_n \leqslant 1$) are not usually locally convex [4, p. 429].

The proof of the theorem is a consequence of the following three lemmas. Here E and F are as above, $f\colon E \to F$ is a continuous map, and K is a compact subset of E.

LEMMA 1. *The sequence (fs_n) converges uniformly to f on K.*

Proof. This follows from the uniform continuity of f on K and the assumption that (s_n) converges uniformly to the identity of E on K.

For a continuous $f\colon E \to F$ we define $f_n = t_n f s_n$. We have

LEMMA 2. *For any neighborhood U of zero in F there is a continuous polynomial $P\colon E \to F$ of finite rank such that $f_n x - Px \in U$ for all $x \in K$.*

Proof. Since E and F are Hausdorff TVSs and s_n and t_n are of finite rank, $s_n(E)$ and $t_n(F)$ are linearly homeomorphic to finite dimensional Euclidean spaces. Since $s_n(K)$ is compact, the classical Weierstrass theorem for Euclidean spaces applied to the restriction $\hat{f}_n$ of f_n to $s_n(E)$ implies there exists a polynomial $\hat{P}\colon s_n(E) \to t_n(F)$ such that $\hat{f}_n x - \hat{P}x \in U$ for all $x \in s_n(K)$. We define an extension P of $\hat{P}$ to E by $Px = \hat{P}s_n x$. Clearly P is a continuously polynomial of finite rank. If $x \in K$, then $f_n x - Px = f_n s_n x - \hat{P}s_n x \in U$.

LEMMA 3. *The sequence (f_n) converges uniformly to f on K.*

Proof. If U is any neighborhood of zero in F, let V be a neighborhood of zero such that $V + V \subseteq U$. Since the sequence (t_n) is equicontinuous, there is a neighborhood of zero W in E such that $t_n(W) \subseteq V$ for all n. By Lemma 1 there is an integer $n_1 \geqslant 0$ such that $fs_n x - fx \in W$ for all $n \geqslant n_1$ and $x \in K$. Finally, since the sequence (t_n) converges uniformly on K to the identity of

F, there is an integer $n_2 \geqslant 0$ such that $t_n fx - fx \in V$ for all $n \geqslant n_2$ and $x \in K$. Hence for all $n \geqslant \max(n_1 n_2)$ and $x \in K$

$$\begin{aligned} f_n x - fx &= t_n(fs_n x - fx) + t_n fx - fx \\ &\subseteq t_n(W) + V \subseteq U. \end{aligned}$$

The theorem clearly follows from Lemmas 2 and 3.

References

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